S.B. 7.10.2008

Geodesics

(M, ∇) manifold with an affine connection.

$\gamma : I \to M$ of class C^2 is called a geodesic for connection ∇ iff

$$\boxed{\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \lambda \cdot \frac{d\gamma}{dt}}$$

for some function $\lambda = \lambda(t)$ along γ.

Recall

local expression
$$\frac{D}{dt}(V) = \left(\frac{dV^\mu}{dt} + \Gamma^\mu{}_{\nu\rho} V^\nu V^\rho\right) X_\rho$$

for $V = V^\mu(t) X_\rho$

vector fields along γ.

$$\frac{d\gamma}{dt} = \frac{dx^\mu}{dt} X_\mu \quad , \quad X_\mu = \frac{\partial}{\partial x^\mu}$$

$\Rightarrow \frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \lambda \frac{d\gamma}{dt}$ can be locally written as

$$\boxed{\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = \lambda \frac{dx^\mu}{dt}}$$

Reparametrization

$$t \to t' = f(t) \quad , \quad \underset{\overset{\parallel}{\dot{f}}}{f'(t)} \neq 0.$$

$$\frac{d}{dt} = \dot{f} \frac{d}{dt'}$$

$$\frac{d^2x^\mu}{dt^2} = \frac{d}{dt}\left(\dot{f} \frac{dx^\mu}{dt'}\right) = \ddot{f} \frac{dx^\mu}{dt'} + \dot{f}^2 \frac{d^2x^\mu}{dt'^2}$$

$$\ddot{f}\frac{dx^\mu}{dt'} + \dot{f}^2\frac{d^2x^\mu}{dt'^2} + \Gamma^\mu{}_{\nu\rho}\dot{f}^2\frac{dx^\nu}{dt'}\frac{dx^\rho}{dt'} = \lambda\dot{f}\frac{dx^\mu}{dt'}$$

$$\frac{d^2x^\mu}{dt'^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt'}\frac{dx^\rho}{dt'} = \frac{\lambda\dot{f}-\ddot{f}}{\dot{f}^2}\frac{dx^\mu}{dt'}$$

$$\boxed{\frac{D}{dt'}\left(\frac{d\gamma}{dt'}\right) = \lambda'\frac{d\gamma}{dt'}, \qquad \lambda' = \frac{\lambda\dot{f}-\ddot{f}}{\dot{f}^2}}$$

<u>Note</u>:

1) Definition does not depend on the choice of paramatrization. If $t \to t' = f(t)$ then $\lambda \to \lambda' = \frac{\lambda\dot{f}-\ddot{f}}{\dot{f}^2}$

2) Given a geodesic with a parametrization for which we have function λ there exists a parametrization for which $\lambda' = 0$ ($\lambda\dot{f} = \ddot{f}$ has always solution).
This parametrization is called <u>affine</u>.

3) Affine parametrization is defined up to an affine transformation $t' \to at' + b$ $\quad a \neq 0$, $a = \text{const}$, $b = \text{const}$.

$$\left(\lambda = 0 \Rightarrow \lambda' = \frac{-\ddot{f}}{\dot{f}^2} = 0 \Rightarrow f = at + b\right)$$

In affine parametrization the geodesic equation is

$$\boxed{\frac{d^2x^\mu}{dt^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = 0}$$

Observe that to determine a geodesic for a connection one needs only to know $\Gamma^\mu{}_{(\nu\rho)}$.

Indeed:

sym.

$$\Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = \Big(\Gamma^\mu{}_{(\nu\rho)} + \underbrace{\cancel{\Gamma^\mu{}_{[\nu\rho]}}}_{\text{antisym}}\Big)\frac{dx^\nu}{dt}\frac{dx^\rho}{dt}$$

$$\Gamma^\mu{}_{\nu\rho} = \underbrace{\tfrac{1}{2}\left(\Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\rho\nu}\right)}_{\Gamma^\mu{}_{(\nu\rho)}} + \underbrace{\tfrac{1}{2}\left(\Gamma^\mu{}_{\nu\rho} - \tfrac{1}{2}\Gamma^\mu{}_{\rho\nu}\right)}_{\Gamma^\mu{}_{[\nu\rho]}}$$

<u>Corollary</u>

There are different connections that have the same geodesics!

(add such antisymmetric (in $\nu\rho$) to $\Gamma^\mu{}_{\nu\rho}$)

<u>How to determine connection?</u>

Recall: $\nabla_\mu X_\nu = \Gamma^\rho{}_{\nu\mu} X_\rho$, $\quad [X_\rho, X_\sigma] = C^\mu{}_{\rho\sigma} X_\mu$

$$T(X_\mu, X_\nu) = \nabla_\mu X_\nu - \nabla_\nu X_\mu - [X_\mu, X_\nu] =$$
$$= \left(\Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu} - C^\rho{}_{\mu\nu}\right) X_\rho = Q^\rho{}_{\mu\nu} X_\rho$$

$$\Rightarrow \quad \Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu} = C^\rho{}_{\mu\nu} + Q^\rho{}_{\mu\nu}$$

$$\boxed{\Gamma^\rho{}_{[\nu\mu]} = \tfrac{1}{2} C^\rho{}_{\mu\nu} + \tfrac{1}{2} Q^\rho{}_{\mu\nu}} \quad \text{(AS)}$$

Corollary

1) Antisymmetric part of the connection is determined by the torsion and the anholonomy coefficients.

$$\boxed{\Gamma^{\rho}{}_{\nu\mu} = \Gamma^{\rho}{}_{(\nu\mu)} + \frac{1}{2} C^{\rho}{}_{\mu\nu} + \frac{1}{2} Q^{\rho}{}_{\mu\nu}}$$

2) Torsionless connection in holonomic frame is symmetric.

(Pseudo) Riemannian connections

(M,g). It is natural to look for a connection ∇ which preserves the metric:

$$\nabla_X g = 0 \qquad \forall X \in \mathfrak{X}(M)$$

$\rightsquigarrow$ $g = g_{\mu\nu}\theta^\mu\theta^\nu$ where $\theta^\mu\theta^\nu = \frac{1}{2}(\theta^\mu\otimes\theta^\nu + \theta^\nu\otimes\theta^\mu)$

$X_\mu \lrcorner \theta^\nu = \delta_\mu{}^\nu$

$$0 = \nabla_\mu g_{\nu\rho} = X_\mu(g_{\nu\rho}) - \Gamma^\alpha{}_{\nu\mu} g_{\alpha\rho} - \Gamma^\alpha{}_{\rho\mu} g_{\nu\alpha} \qquad | \, \theta^\mu$$

$$\boxed{0 = D g_{\nu\rho} = d g_{\nu\rho} - g_{\rho\alpha}\Gamma^\alpha{}_\nu - g_{\nu\alpha}\Gamma^\alpha{}_\rho} \quad (MC)$$

Digression

Assumption about the nondegeneracy of g

$$g(X,Y)=0 \quad \forall Y \quad \Rightarrow \quad X=0 \quad \text{means that}$$

at every point the map:

$$T_pM \ni X_p \longrightarrow g_p(X_p, \cdot) \in T_p^*M$$

is isomorphism of vector spaces T_pM and T_p^*M.

Using the metric (even is ΨRiemannian case!) we can identify TM and T^*M.

In the old tensorial language:

(•) V^μ – coefficients of a vector field V

$$V^\mu \xmapsto{g_{\mu\nu}} g_{\mu\nu}V^\nu = V_\nu \leftarrow \text{coefficients of a 1-form}$$

(••) λ_μ – coefficients of a 1-form λ

$g_{\mu\nu}$ is invertible $\Rightarrow$ $g^{\alpha\beta}$ s.t. $g^{\alpha\beta}g_{\beta\mu} = \delta^\alpha_\mu$ is uniquely defined by g.

$$\lambda_\mu \xmapsto{g^{\mu\nu}} g^{\mu\nu}\lambda_\nu = \lambda^\nu \leftarrow \text{coefficients of a vector field.}$$

Our condition (MC) for a connection that preserves the metric

$$dg_{\nu\rho} = \Gamma_{\rho\nu} + \Gamma_{\nu\rho} \qquad \text{or}$$

$$\boxed{\Gamma_{\rho\nu\mu} + \Gamma_{\nu\rho\mu} = X_\mu(g_{\nu\rho})} \qquad (1)$$

We also have:

$$\boxed{\Gamma_{\rho\nu\mu} - \Gamma_{\rho\mu\nu} = c^\rho_{\mu\nu} + Q^\rho_{\mu\nu}} \qquad (2)$$

(1) - (2):

$$\Gamma_{\nu\rho\mu} + \Gamma_{\rho\mu\nu} = X_\mu(g_{\nu\rho}) - c^\rho_{\mu\nu} - Q^\rho_{\mu\nu} =: H_{\nu\rho\mu}$$

$$\begin{array}{c|l} & \underline{\Gamma_{\nu\rho\mu}} + \Gamma_{\rho\mu\nu} = H_{\nu\rho\mu} \\ + & \Gamma_{\mu\nu\rho} + \underline{\Gamma_{\nu\rho\mu}} = H_{\mu\nu\rho} \\ - & \Gamma_{\rho\mu\nu} + \Gamma_{\mu\nu\rho} = H_{\rho\mu\nu} \end{array}$$

$$2\Gamma_{\nu\rho\mu} = H_{\nu\rho\mu} + H_{\mu\nu\rho} - H_{\rho\mu\nu}$$

$$\left\{ \begin{array}{l} \boxed{\Gamma_{\nu\rho\mu} = \frac{1}{2}\left(H_{\nu\rho\mu} + H_{\mu\nu\rho} - H_{\rho\mu\nu}\right)} \quad \text{where} \\ \boxed{H_{\nu\rho\mu} = X_\mu(g_{\nu\rho}) - c^\rho_{\mu\nu} - Q^\rho_{\mu\nu}} \end{array} \right.$$